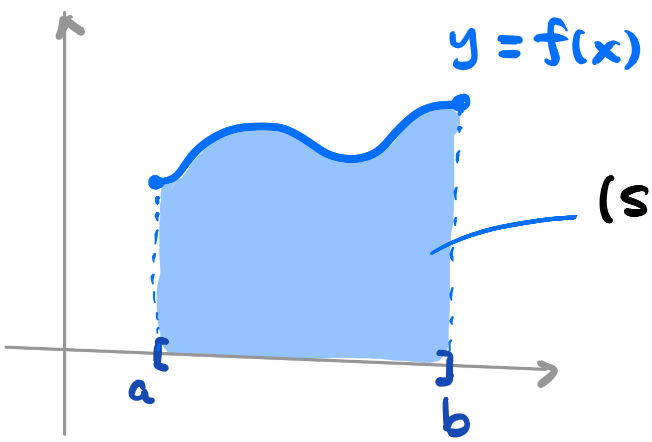
MATH 2028 - Multiple Integrals on rectangles

GOAL: Define the (Riemann) integral $\int_R f$ for a multi-variable function $f: R \to \mathbb{R}$ defined on a rectangle $R \subseteq \mathbb{R}^n$ & study their basic properties

Recall: Integration of $f: [a,b] \to \mathbb{R}$



(signed) area under curve

$$= \int_a^b f(x)\,dx$$

Idea: Approximation by "rectangles" from above and below, then take a "limit".

Remark: This idea works in any dimension, as long as we know how to define the "volume" of a "rectangle" in $\mathbb{R}^n$.

Def$^{\underline{n}}$: A subset $R \subseteq \mathbb{R}^n$ is a **rectangle** if

$$R = [a_1, b_1] \times [a_2, b_2] \times \cdots \times [a_n, b_n]$$

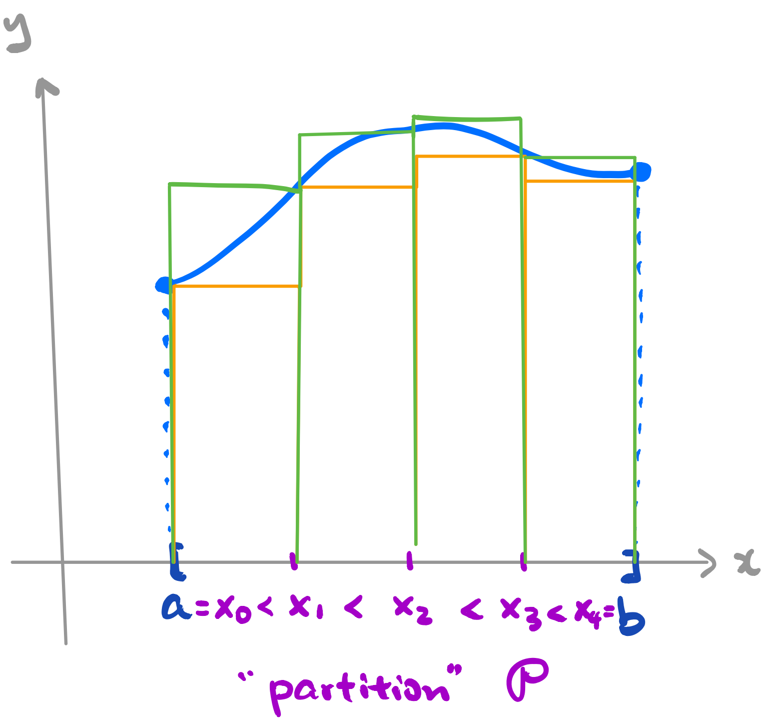
for some real numbers $a_i < b_i$, $i = 1, \ldots, n$.

The **volume** of a rectangle R is define as:

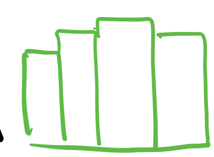
$$\mathrm{Vol}(R) := (b_1 - a_1)(b_2 - a_2) \cdots (b_n - a_n)$$

To carry out the approximations, we need to "partite" a rectangle $R \subseteq \mathbb{R}^n$ into small sub-rectangles. We first do it for $n = 1$, where the notations are much simpler.

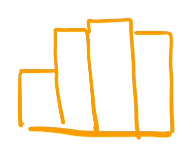
Graph of $f : [a, b] \to \mathbb{R}$



"Upper sum"

$\mathcal{U}(f, \mathcal{P}) = \text{Area}\Big($  $\Big)$

"Lower sum"

$L(f, \mathcal{P}) = \text{Area}\Big($  $\Big)$

Note: For ANY partition $\mathcal{P}$.

$$L(f, \mathcal{P}) \leq \int_a^b f \leq \mathcal{U}(f, \mathcal{P})$$

Def$^{\underline{n}}$: Let $R = [a_1, b_1] \times \dots \times [a_n, b_n] \subseteq \mathbb{R}^n$.

Suppose there is a partition of each $[a_i, b_i]$, $i = 1, \dots, n$, into sub-intevals:

$$a_i = x_{i,0} < x_{i,1} < \dots < x_{i,k_i} = b_i$$

Then, we say that $\mathcal{P} = \{R_{i_1 i_2 \cdots i_n}\}$ is a partition of R into sub-rectangles

$$R_{i_1 i_2 \cdots i_n} := [x_{1,i_1-1}, x_{1,i_1}] \times [x_{2,i_2-1}, x_{2,i_2}] \times \dots \times \dots \times [x_{n,i_n-1}, x_{n,i_n}]$$

Given any <u>bounded</u> $f : R \longrightarrow \mathbb{R}$, we define the upper sum and lower sum of f w.r.t. $\mathcal{P}$ respectively by:

$$\mathcal{U}(f, \mathcal{P}) := \sum_{i_1, \dots, i_n} \sup_{x \in R_{i_1 \cdots i_n}} f(x) \cdot \mathrm{Vol}(R_{i_1 \cdots i_n})$$

$$\geq$$

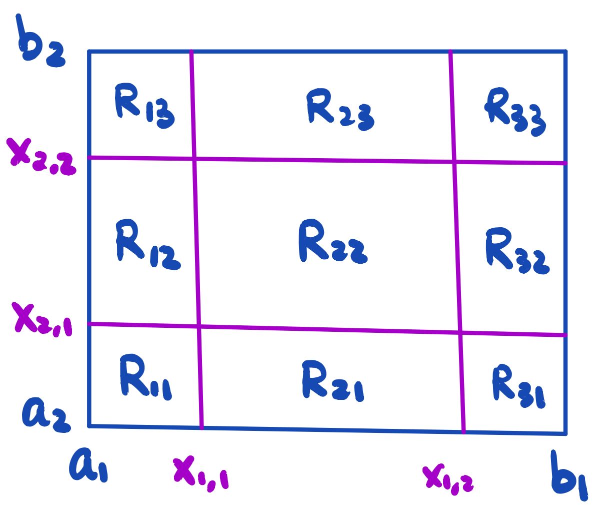
$$L(f, \mathcal{P}) := \sum_{i_1, \dots, i_n} \inf_{x \in R_{i_1 \cdots i_n}} f(x) \cdot \mathrm{Vol}(R_{i_1 \cdots i_n})$$

Remark: The sup & inf exist since f is bounded.

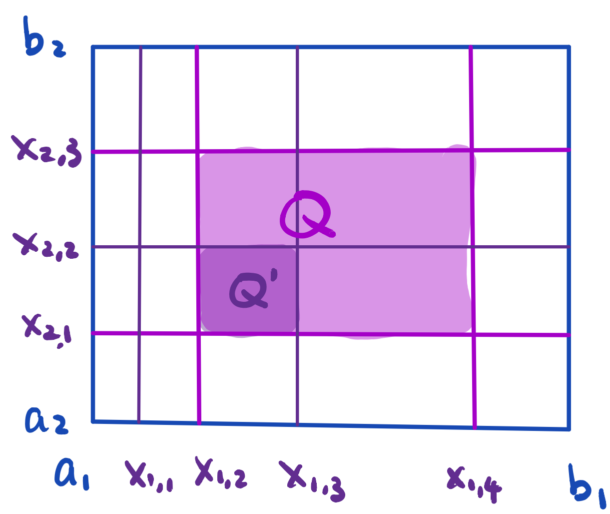
Note that the upper and lower sums of f depends on the partition $\mathcal{P}$. The following notion is useful when one wants to compare two different partitions.

<u>Defn</u>: Let $\mathcal{P}$ be a partition of a rectangle R.
A **refinement** of $\mathcal{P}$ is another partition $\mathcal{P}'$ of R s.t. $\forall\ Q' \in \mathcal{P}',\ \exists\ Q \in \mathcal{P}$ s.t. $Q' \subseteq Q$.

Example:



a partition $\mathcal{P}$ of
$R = [a_1, b_1] \times [a_2, b_2]$



a refinement $\mathcal{P}'$ of the partition $\mathcal{P}$

Easy Fact: Given any two partitions $\mathcal{P}, \mathcal{P}'$ of R, $\exists$ a partition $\mathcal{P}''$ of R which is a refinement for both $\mathcal{P}$ and $\mathcal{P}'$. We call such $\mathcal{P}''$ a common refinement of $\mathcal{P}$ and $\mathcal{P}'$.

Remark: The set of partitions of R forms a "partially ordered set"

Lemma: If $\mathcal{P}'$ is a refinement of $\mathcal{P}$, then

$$\boxed{L(f,\mathcal{P}) \leq L(f,\mathcal{P}') \leq U(f,\mathcal{P}') \leq U(f,\mathcal{P})}$$

Proof: Write $\mathcal{P} = \{Q\}$ and $\mathcal{P}' = \{Q'\}$.

From definition, for each rectangle $Q \in \mathcal{P}$, $\exists\ Q_1', \dots, Q_r' \in \mathcal{P}'$ s.t. $Q = Q_1' \cup \dots \cup Q_r'$

By def$^{\underline{n}}$ of infimum, we have for $i = 1, \dots, r$,

$$\inf_{x \in Q} f(x) \cdot \mathrm{Vol}(Q_i') \leq \inf_{x \in Q_i'} f(x) \cdot \mathrm{Vol}(Q_i')$$

Summing over i, as $\mathrm{Vol}(Q) = \sum_{i=1}^{r} \mathrm{Vol}(Q_i')$,

$$\inf_{x \in Q} f(x) \cdot \mathrm{Vol}(Q) \leq \sum_{i=1}^{r} \inf_{x \in Q_i'} f(x) \cdot \mathrm{Vol}(Q_i')$$

Summing over all $Q \in \mathcal{P}$, we obtain

$$L(f, \mathcal{P}) \le L(f, \mathcal{P}')$$

Similarly, $U(f, \mathcal{P}) \ge U(f, \mathcal{P}')$ since we have

$\sup_{x \in Q} f(x) \ge \sup_{x \in Q_i'} f(x)$.

$\blacksquare$

<u>Lemma:</u> For any partitions $\mathcal{P}$, $\mathcal{P}'$ of R,

$$\boxed{L(f, \mathcal{P}) \le U(f, \mathcal{P}')}$$

<u>Proof:</u> Take any common refinement $\mathcal{P}''$ of $\mathcal{P}$ and $\mathcal{P}'$. By previous lemma,

$$L(f, \mathcal{P}) \le L(f, \mathcal{P}'') \le U(f, \mathcal{P}'') \le U(f, \mathcal{P}').$$

$\blacksquare$

This lemma implies that for any bdd $f : R \to \mathbb{R}$, we always have

$$\sup_{\mathcal{P}} L(f, \mathcal{P}) \le \inf_{\mathcal{P}} U(f, \mathcal{P})$$

We are mostly interested in the "equality" case.

Def$^{\underline{n}}$: A bounded function $f: R \to \mathbb{R}$ is (Riemann) integrable on a rectangle $R \subseteq \mathbb{R}^n$ if

$$\sup_{\mathcal{P}} L(f, \mathcal{P}) = \inf_{\mathcal{P}} U(f, \mathcal{P})$$

The common value, denoted by $\int_R f \, dV$, is called the integral of f over R.

Example: (Constant functions)

Let $f: R \to \mathbb{R}$ be a constant function defined on a rectangle $R \subseteq \mathbb{R}^n$, i.e., $\exists \, c \in \mathbb{R}$ s.t.

$$f(x) \equiv c \quad \forall x \in R$$

Then, f is integrable on R and

$$\int_R f \, dV = c \cdot \text{Vol}(R)$$

Proof: For ANY partition $\mathcal{P}$ of R, we have

$$L(f, \mathcal{P}) = \sum_{Q \in \mathcal{P}} \overbrace{\inf_{x \in Q} f(x)}^{= c} \cdot \text{Vol}(Q)$$

$$= c \cdot \sum_{Q \in \mathcal{P}} \text{Vol}(Q) = c \cdot \text{Vol}(R)$$

Similarly, $\mathcal{U}(f, \mathcal{P}) = c \cdot \mathrm{Vol}(R)$.

Since $L(f, \mathcal{P}) = c \cdot \mathrm{Vol}(R) = \mathcal{U}(f, \mathcal{P})$

for <u>ALL</u> partition $\mathcal{P}$, by def^{n}, f is

integrable on R with $\int_R f \, dV = c \cdot \mathrm{Vol}(R)$.

$\blacksquare$

<u>Example</u>: (A nowhere continuous function)

Let $f : [a,b] \to \mathbb{R}$ be a bdd function s.t.

$$f(x) = \begin{cases} 1, & \text{if } x \in \mathbb{Q} \cap [a,b] \\ 0, & \text{if } x \in [a,b] \setminus \mathbb{Q} \end{cases}$$

Then, f is <u>NOT</u> integrable on $[a,b]$.

<u>Proof</u>: Recall that $\mathbb{Q}$ and $\mathbb{Q}^c$ are both dense in $\mathbb{R}$. For <u>ANY</u> partition $\mathcal{P}$ of $[a,b]$,

$$L(f, \mathcal{P}) = \sum_{Q \in \mathcal{P}} \overbrace{\inf_{x \in Q} f(x)}^{=0} \cdot \mathrm{Vol}(Q) = 0$$

$$\mathcal{U}(f, \mathcal{P}) = \sum_{Q \in \mathcal{P}} \overbrace{\sup_{x \in Q} f(x)}^{=1} \cdot \mathrm{Vol}(Q) = b - a$$

Therefore,

$$\sup_{\mathcal{P}} L(f,\mathcal{P}) = 0 < b-a = \inf_{\mathcal{P}} U(f,\mathcal{P})$$

and hence f is **NOT** integrable on $[a,b]$. $\blacksquare$

Ex: Find a bdd function $f : R \subseteq \mathbb{R}^2 \to \mathbb{R}$ which is NOT integrable over R.